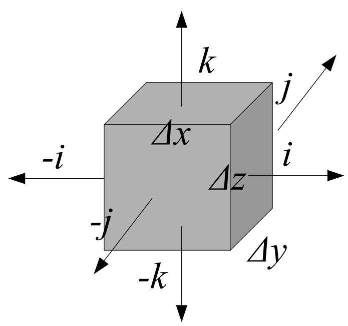
GEOMETRIC INTERPRETATION OF DIVERGENCE

Suppose we have a vector field

$$\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$$

where $\mathbf{f}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$ — and we want to measure how much it "spreads out" from a tiny cube of dimensions Δx, Δy, and Δz — per the cube's *volume*:



This cube is so small we can assume the vector-field is *constant* over the faces of the cube.

To find out how much of this field *leaves* a face of the cube (say the top face) we form the dot-product $\mathbf{f}(x, y, z + \Delta z) \bullet \mathbf{k} \cdot \Delta x \Delta y = f_3(x, y, z + \Delta z)\Delta x \Delta y$ — since $\mathbf{i} \bullet \mathbf{k} = \mathbf{j} \bullet \mathbf{k} = 0$. Here, the factor $\Delta x \Delta y$ is the *area* of that face.

The same computation on the opposite face gives $\mathbf{f}(x, y, z) \bullet (-\mathbf{k}) \cdot \Delta x \Delta y = -f_3(x, y, z)\Delta x \Delta y$. The total outflow (from all 6 faces) is:

$$\begin{aligned} f_3(x, y, z + \Delta z)\Delta x \Delta y &- f_3(x, y, z)\Delta x \Delta y \\ +f_2(x, y + \Delta y, z)\Delta x \Delta z &- f_2(x, y, z)\Delta x \Delta z \\ +f_1(x + \Delta x, y, z)\Delta y \Delta z &- f_1(x, y, z)\Delta y \Delta z \\ &= \Delta x \Delta y (f_3(x, y, z + \Delta z) - f_3(x, y, z)) \\ &+ \Delta x \Delta z (f_2(x, y + \Delta y, z) - f_2(x, y, z)) \\ &+ \Delta y \Delta z (f_1(x + \Delta x, y, z) - f_1(x, y, z)) \end{aligned}$$

Now we divide by the volume of the cube, $\Delta x \Delta y \Delta z$, to get the outflow *per volume*:

$$\frac{f_3(x, y, z + \Delta z) - f_3(x, y, z)}{\Delta z} + \frac{f_2(x, y + \Delta y, z) - f_2(x, y, z)}{\Delta y} + \frac{f_1(x + \Delta x, y, z) - f_1(x, y, z)}{\Delta x}$$

If we take the limit as Δx, Δy, Δz all go to 0, we get

$$\frac{\partial f_3}{\partial z} + \frac{\partial f_2}{\partial y} + \frac{\partial f_1}{\partial x} = \operatorname{div} \mathbf{f} = \nabla \bullet \mathbf{f}$$

Note that $\nabla \bullet \mathbf{f}$ is a (very common) alternate notation for $\operatorname{div} \mathbf{f}$, where we think of ∇ as the "vector"

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

This can be applied to the *Continuity Equation* for fluid flow:

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{v}) = 0$$

where $\rho(x, y, z)$ is the *density* of the fluid and $\mathbf{v}$ is the *velocity* (vector field).