EIGENVALUES AND EIGENVECTORS

1. Definition

They are defined in terms of each other. Let A be an $n \times n$ matrix. A vector $v \neq 0$ is an *eigenvector* of A with *eigenvalue* λ if the equation

$$Av = \lambda v$$

is satisfied.

Note that eigenvectors are *not* uniquely defined:

If v is an eigenvector then *any scalar multiple* of v is also an eigenvector. In fact, if u and v are two eigenvectors for the same eigenvalue λ, then *any linear combination* $a \cdot u + b \cdot v$ is *also* an eigenvector with eigenvalue λ.

2. Applications of eigenvectors

You might wonder why anyone would be interested in eigenvectors and eigenvalues. Eigenvectors are vectors toward which the matrix A behaves like a *scalar* — namely the scalar λ (the corresponding eigenvalue).

If we could find a basis consisting of eigenvectors, A would become a diagonal matrix in this basis: its action on basis-vectors would only be to multiply each of them by a scalar — which is what a diagonal matrix does.

Diagonal matrices are interesting because the are easy to work with — they behave like *scalars* when you add or multiply them.

One can find closed-form expressions for A^n or even A^x where x is a variable. The value of e^A is useful for solving linear systems of differential equations.

3. Computing eigenvalues and eigenvectors

We return to the defining equation

$$Av = \lambda v$$

and consider what it is saying. We get

$$\begin{aligned} Av &= \lambda v \\ Av &= \lambda \cdot Iv \\ Av - \lambda \cdot Iv &= 0 \\ (A - \lambda \cdot I)v &= 0 \end{aligned}$$

where I is the identity matrix. Since $v \neq 0$ (by definition), it follows that the matrix $A - \lambda \cdot I$ is not invertible when λ is an eigenvalue, so

$$\det(A - \lambda \cdot I) = 0$$

Definition 1. Regard λ as a variable and define $p(\lambda) = \det(A - \lambda \cdot I)$ — this is called the *characteristic polynomial* of A. Its roots are the *eigenvalues* of A.

Our method for computing eigenvalues and eigenvectors is:

(1) compute the *characteristic polynomial*
(2) find its *roots* — these are the eigenvalues
(3) for each of these eigenvalues, compute the corresponding eigenvectors. Sometimes there may be many different eigenvectors. In general, the eigenvectors corresponding to an eigenvalue form a subspace called the eigenspace corresponding to that eigenvalue.

Example:
Let

$$A = \begin{bmatrix} -17 & 42 \\ -9 & 22 \end{bmatrix}$$

To compute the eigenvalues of A compute the characteristic polynomial

$$p(\lambda) = \det(A - \lambda \cdot I) = \det \begin{bmatrix} -17-\lambda & 42 \\ -9 & 22-\lambda \end{bmatrix} = \lambda^2 - 5\lambda + 4$$

The roots of this polynomial are 1 and 4. For the eigenvalue $\lambda = 1$ the corresponding eigenvector is the solution of

$$\begin{bmatrix} -17-1 & 42 \\ -9 & 22-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -18 & 42 \\ -9 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} v = 0$$

so $x_2 = \frac{21}{9} x_1 = \frac{7}{3} x_1$ is a solution and we get an eigenvector

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

For the eigenvalue 4 we must solve

$$\begin{bmatrix} -17-4 & 42 \\ -9 & 22-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -21 & 42 \\ -9 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} v = 0$$

for which $x_1 = 2x_2$ is a solution, and this gives the eigenvector

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

To summarize, we have

- $\lambda = 1$, $v = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$
- $\lambda = 4$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

4. Applications: changing bases

A matrix is an expression of a linear transformation, *with respect to some basis:* if T is a linear transformation and $\{e_i\}$ is a basis, the matrix representing T is

$$A = \begin{bmatrix} Te_1 & \cdots & Te_n \end{bmatrix}$$

where the vectors Te_i are written in the $\{e_i\}$ basis. If $\{b_j\}$ is a new basis for the same vector space, then the matrix

$$B = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}$$

converts from the b-basis back to the e-basis. Its inverse converts in the opposite direction. If A is a matrix in the e-basis, the form of A in the b-basis can be computed:

$$\begin{array}{ccc} e-\text{basis} & \xrightarrow{A} & e-\text{basis} \\ B\uparrow & & \uparrow B \\ b-\text{basis} & \xrightarrow[D]{} & b-\text{basis} \end{array}$$

where D is the same linear transformation as A, but in the b-basis. To reverse any arrow, take the inverse of the matrix labeling it. For instance, we get

$$\begin{array}{ccc} e-\text{basis} & \xrightarrow{A} & e-\text{basis} \\ B\uparrow & & \downarrow B^{-1} \\ b-\text{basis} & \xrightarrow[D]{} & b-\text{basis} \end{array}$$

so the path through the diagram implies that

$$D = B^{-1}AB$$

and

$$A = BDB^{-1}$$

Now we apply this to the example in the last section. Recall that

- $A = \begin{bmatrix} -17 & 42 \\ -9 & 22 \end{bmatrix}$
- $\lambda = 1$, $v = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$
- $\lambda = 4$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The eigenvectors form a basis for $\mathbb{R}^2$ so

$$B = \begin{bmatrix} 3 & 2 \\ 7 & 1 \end{bmatrix}$$

and, in the eigen-basis, A becomes the diagonal matrix

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Since D is the same linear transformation as A, with respect to a different basis, we have

$$A^n = BD^nB^{-1}$$

and we get a *closed-form formula* for A^n:

$$A^n = \begin{bmatrix} 7 - 6\cdot 4^n & -14 + 14\cdot 4^n \\ 3 - 3\cdot 4^n & -6 + 7\cdot 4^n \end{bmatrix}$$

If you set $n = 0$ this formula becomes the identity matrix, and if you set $n = 1$ it becomes A.

This formula is even valid for *non-integral* values of n:

$$Z = A^{1/2} = \begin{bmatrix} -5 & 14 \\ -3 & 8 \end{bmatrix}$$

It is easy to verify that $Z^2 = A$.

We can even compute other functions of matrices. Define

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

— the standard power-series for e^x. We can plug a matrix, A, into this series (replacing 1 by the suitable identity matrix) and it will converge — giving a meaning to e^A.

Computing the value of this matrix can be difficult — unless we convert A to a *diagonal matrix:*

$$e^D = \begin{bmatrix} 1 + 1 + 1/2! \cdots & 0 \\ 0 & 1 + 4 + 4^2/2! \cdots \end{bmatrix} = \begin{bmatrix} e^1 & 0 \\ 0 & e^4 \end{bmatrix}$$

Then we compute

$$e^A = Be^D B^{-1} = \begin{bmatrix} 7\,\mathrm{e}^1 - 6\,\mathrm{e}^4 & -14\,\mathrm{e}^1 + 14\,\mathrm{e}^4 \\ 3\,\mathrm{e}^1 - 3\,\mathrm{e}^4 & -6\,\mathrm{e}^1 + 7\,\mathrm{e}^4 \end{bmatrix}$$