# EXTERIOR PRODUCTS AND DETERMINANTS

This gives a slightly exotic way of computing determinants using a concept called exterior products.

Define a product operation, $\wedge$ — called the *exterior product,* via

(1) $\wedge$ is *anti-commutative* — $u \wedge v = -v \wedge u$. Note that this implies that $u \wedge u = 0$ for any vector $u$.
(2) $(c \cdot u) \wedge v = c \cdot (u \wedge v)$
(3) It is distributive in both variables: $(u_1+u_2)\wedge v = u_1\wedge v+u_2\wedge v$, $u\wedge(v_1+v_2) = u \wedge v_1 + u \wedge v_2$

For the time being, forget about what this product "means" — it turns out to have a geometric interpretation as a kind of generalization of $\times$-product.

Using the three rules above, it is possible to formally evaluate $\wedge$-products of vectors in terms of $\wedge$-products of basis vectors:

Suppose

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and suppose $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$. Then

$$\begin{aligned} u &= e_1 + 2e_2 + 3e_3 \\ v &= 3e_1 - 2e_2 + 5e_3 \end{aligned}$$

and

$$\begin{aligned} u \wedge v &= 3e_1 \wedge e_1 - 2e_1 \wedge e_2 + 5e_1 \wedge e_3 \\ &\quad +6e_2 \wedge e_1 - 4e_2 \wedge e_2 + 10e_2 \wedge e_3 \\ &\quad +9e_3 \wedge e_1 - 6e_3 \wedge e_2 + 15e_3 \wedge e_3 \\ &= -2e_1 \wedge e_2 + 5e_1 \wedge e_3 \\ &\quad +6e_2 \wedge e_1 + 10e_2 \wedge e_3 \\ &\quad +9e_3 \wedge e_1 - 6e_3 \wedge e_2 \\ &= -8e_1 \wedge e_2 - 4e_1 \wedge e_3 \\ &\quad +16e_2 \wedge e_3 \end{aligned}$$

Quantities like $u\wedge v$ are called *2-tensor densities.* Just as a vector is a magnitude and a *direction*, a 2-tensor density is a magnitude and a *plane.*

In like fashion, we can define 3-tensor densities:

If $w = e_1 + e_2 + e_3$ we can compute the exterior product

$$\begin{aligned} u \wedge v \wedge w &= (-8e_1 \wedge e_2 - 4e_1 \wedge e_3 + 16e_2 \wedge e_3) \wedge (e_1 + e_2 + e_3) \\ &= \quad -8e_1 \wedge e_2 \wedge e_1 - 4e_1 \wedge e_3 \wedge e_1 + 16e_2 \wedge e_3 \wedge e_1 \\ &\quad\quad -8e_1 \wedge e_2 \wedge e_2 - 4e_1 \wedge e_3 \wedge e_2 + 16e_2 \wedge e_3 \wedge e_2 \\ &\quad\quad -8e_1 \wedge e_2 \wedge e_3 - 4e_1 \wedge e_3 \wedge e_3 + 16e_2 \wedge e_3 \wedge e_3 \\ &= 16e_2 \wedge e_3 \wedge e_1 - 4e_1 \wedge e_3 \wedge e_2 - 8e_1 \wedge e_2 \wedge e_3 \\ &= 12e_1 \wedge e_2 \wedge e_3 \end{aligned}$$

Note, that with 3 basis vectors $e_1, e_2, e_3$ all 3-fold exterior products are multiples of $e_1 \wedge e_2 \wedge e_3$ and all exterior products of 4 or more terms *vanish*.

Tensors are used to describe phenomena that naturally act along a plane or higher-dimensional region. For instance the stess inside a cube of metal that is being twisted on an axis naturally acts along a plane. If the stress is stong enough to rip apart the metal cube, it split along this plane. This is best described with a *2-tensor* (and is where the term "tensor" came from) rather than a *vector* (which is a 1-tensor).

**Definition 1.** If $V$ is a vector space (i.e., $\mathbb{R}^n$, for some value of $n$, let $\Lambda^k V$ represent the set of all $k$-fold wedge-products of vectors from $V$.

If $\{e_i\}$ is a basis for $V$ then the wedge-products, $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$, form a basis for $\Lambda^k V$, where $\{i_1, \dots, i_k\}$ runs over all *increasing* sequences of subscripts, i.e., $i_1 < i_2 < \cdots < i_k$. It follows that, if $V$ is $n$-dimensional, $\Lambda^k V$ is

$$\binom{n}{k}$$

dimensional. In particular $\Lambda^k V = 0$ if $k > n$, and

(1) $\Lambda^0 V = \mathbb{R}$, the scalars
(2) $\Lambda^1 V = V$

Here's our exotic way of defining the determinant:

**Proposition 2.** *If $A$ is an $n \times n$ matrix with column-vectors $v_1, \dots, v_n$, written as linear combinations of basis-elements $\{e_i\}$, then*

$$v_1 \wedge \cdots \wedge v_n = \det A \cdot e_1 \wedge \cdots \wedge e_n$$

*Proof.* Since $\det A = \det A^{tr}$ — the transpose of $A$, let's transpose $A$ so we get a matrix for which the $v_i$ are *row*-vectors, and compute its determinant. If we add a multiple of one row to another, the value of the wedge-product is *unchanged*:

$$\begin{aligned} &v_1 \wedge \cdots \wedge (v_i + kv_j) \wedge \cdots \wedge v_n = \\ &\quad v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_n + v_1 \wedge \cdots \wedge kv_j \wedge \cdots \wedge v_n = \\ &\quad v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_n + kv_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_n = \\ &\quad v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_n + kv_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_j \wedge \cdots \wedge v_n = \\ &\quad v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_n \end{aligned}$$

since $v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_j \wedge \cdots \wedge v_n = 0$ because $v_j$ occurs twice. It follows that we can perform any elementary row operations we please on the vectors $\{v_i\}$ without changing the value of

$$v_1 \wedge \cdots \wedge v_n$$

Interchanging any two of the $\{v_i\}$ multiplies the wedge product by $-1$ — just as it does with the determinant.

It follows that we can perform elementary row operations that put the matrix in echelon form, $\bar{A}$, without changing the wedge-product — i.e., we can replace the $\{v_i\}$ by rows $\{w_i\}$ where $w_n = c_n \cdot e_n$, $w_{n-1} = c_{n-1} \cdot e_{n-1} + d \cdot e_n$, and so on, so $w_k$ is a linear combination of $\{e_k, e_{k+1}, \dots, e_n\}$ — and that

$$v_1 \wedge \cdots \wedge v_n = w_1 \wedge \cdots \wedge w_n$$

The fact that the $e_i$ annihilate $e_k$ with $j = i$ implies that $w_1 \wedge \cdots \wedge w_n$ is just the product of the elements on the *diagonal* of the matrix $\bar{A}$. But this product of diagonal elements is equal to the determinant. □

Note that $\binom{n}{k} = \binom{n}{n-k}$. This implies that $\Lambda^k V = \Lambda^{n-k} V$, as vector-spaces. The actual mapping between them is defined by

$$e_{i_1} \wedge \dots e_{i_k} \mapsto c \cdot e_{j_1} \wedge \cdots \wedge e_{j_{n-k}} \tag{0.1}$$

where $\{j_1, \dots j_{n-k}\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$ and $c = \pm 1$. The actual value of $c$ is given by

$$(e_{i_1} \wedge \dots e_{i_k}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}) = c \cdot e_1 \wedge \cdots \wedge e_n$$

Here, we sort the factors of $(e_{i_1} \wedge \dots e_{i_k}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_{n-k}})$ until they are in *ascending order* — and each swap multiplies the whole term by $-1$.

If $V = \mathbb{R}^3$ we have $\Lambda^2 V = \Lambda^1 V = V$ — i. e., *two-tensors* can be regarded as *vectors*. This is because, (in $\mathbb{R}^3$) there's a 1-1 correspondence between planes and the lines perpendicular to them.

This gives the geometric meaning of cross-products (and why one must be in three dimensions for them to make sense): The cross-product of vectors $v$ and $w$ is $v \wedge w$ mapped to a vector under the correspondence in equation 0.1.