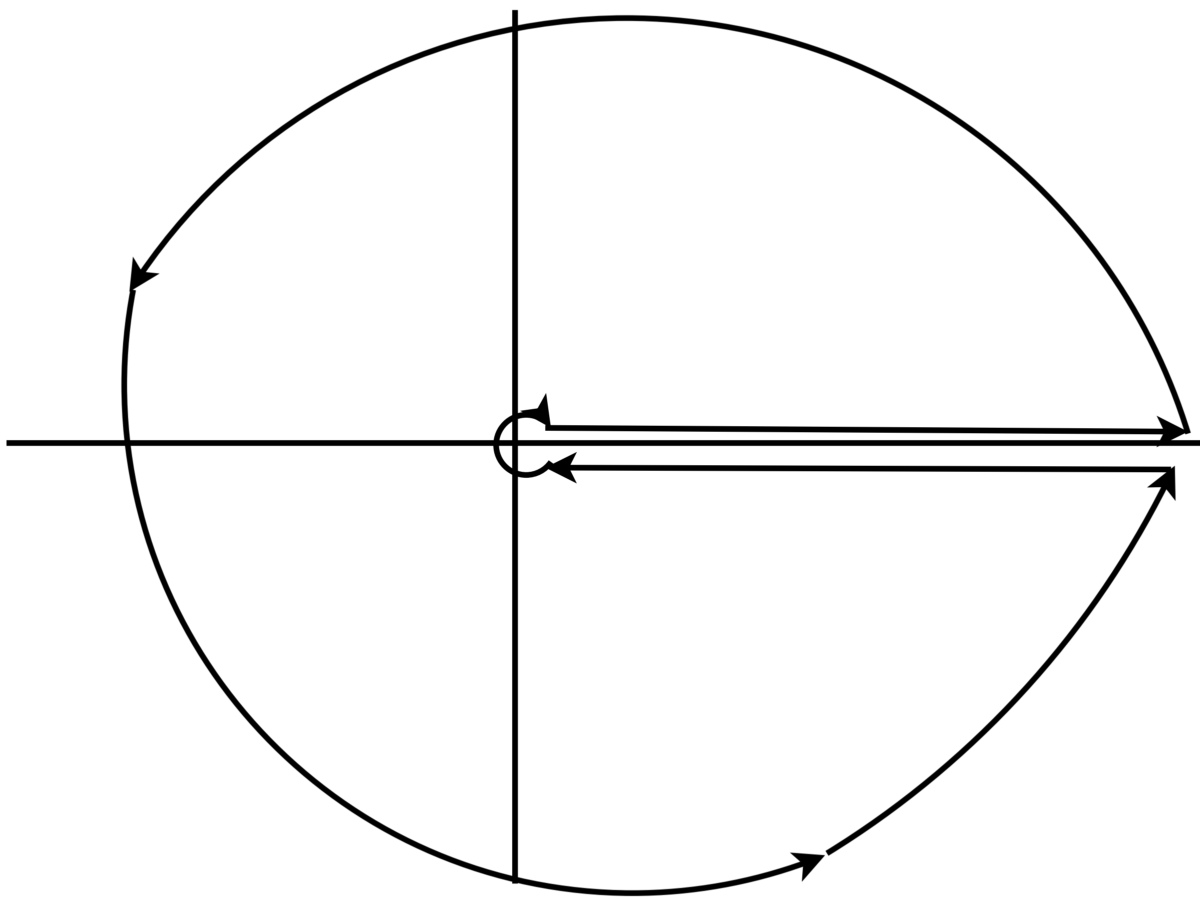
PRACTICE PROLEMS

1. Calculus of Residues

(1) Compute\int

$$I = \int_0^\infty x^{1/2} \frac{1}{1+x+x^2}\, dx$$

Solution: Use the contour, C:



We get

$$\int_C z^{1/2} \frac{1}{1+z+z^2}\, dx = I - e^{\pi i} I = 2\pi i \sum \text{Residues in } C$$

and we can calculate the residues at the poles of

$$\frac{1}{1+z+z^2}$$

which are $e^{2\pi i/3}$ and $e^{4\pi i/3}$. We might also conclude that $e^{4\pi i/3} = e^{-2\pi i/3}$ — which is technically true — but you must *cross the branch-cut* to get this (i.e., it involves wrapping around 2π). Since we aren't allowed to violate the branch cut — we can *only* use $e^{4\pi i/3}$ in our computations and *not* $e^{-2\pi i/3}$. The residues are

- at $z = e^{2\pi i/3}$, we get
$$(e^{2\pi i/3})^{1/2} \lim_{z\to e^{2\pi i/3}} \frac{z - e^{2\pi i/3}}{1+z+z^2} = e^{\pi i/3}\frac{1}{1+2e^{2\pi i/3}} = \frac{-ie^{\pi i/3}}{\sqrt{3}}$$
- at $z = e^{4\pi i/3}$, we get
$$(e^{4\pi i/3})^{1/2} \lim_{z\to e^{4\pi i/3}} \frac{z - e^{4\pi i/3}}{1+z+z^2} = e^{2\pi i/3}\frac{1}{1+2e^{4\pi i/3}} = \frac{ie^{2\pi i/3}}{\sqrt{3}}$$

and the total of the residues is
$$\frac{i}{\sqrt{3}}(-e^{\pi i/3} + e^{2\pi i/3}) = \frac{i}{\sqrt{3}}(-\cos(\pi/3) - i(\sin\pi/3) + \cos(2\pi/3) + i\sin(2\pi/3))$$
Since $\sin(2\pi/3) = \sin(\pi/3)$ and $\cos(2\pi/3) = -\cos(\pi/3)$, the sines cancel out and the cosines add to give
$$-\frac{2i\cos(\pi/3)}{\sqrt{3}} = -\frac{i}{\sqrt{3}}$$
We get
$$I(1-e^{\pi i}) = 2I = 2\pi i \cdot (-\frac{i}{\sqrt{3}}) = \frac{2\pi}{\sqrt{3}}$$
so we finally get the solution to our problem
$$\int_0^\infty x^{1/2}\frac{1}{1+x+x^2}\,dx = \frac{\pi}{\sqrt{3}}$$

(2) Compute
$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^6}\,dx$$
We set
$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^6}\,dx = \int_C \frac{z^2}{1+z^6}\,dz = 2\pi i \sum \text{Residues in upper half-plane}$$
where C is the upper half-circle. The poles of $\frac{z^2}{1+z^6}$ are the 6^{th} roots of -1. These are
$$e^{\pi i/6}, e^{3\pi i/6} = i, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6} = -i, e^{11\pi i/6}$$
Only the first three will concern us (since they lie in the upper half-plane). The residues at these points are

- At $z = e^{\pi i/6}$, we get
$$(e^{\pi i/6})^2 \lim_{z\to e^{\pi i/6}} \frac{z - e^{\pi i/6}}{1+z^6}$$
and l'Hopital's rule gives
$$e^{\pi i/3} \lim_{z\to e^{\pi i/6}} \frac{1}{6z^5} = e^{\pi i/3}\frac{e^{-5\pi i/6}}{6} = \frac{e^{-3\pi i/6}}{6} = -\frac{i}{6}$$

- At $z = i$, we get

$$(i)^2 \lim_{z \to i} \frac{z - i}{1 + z^6} = -\frac{1}{6i^5} = -\frac{1}{6i} = \frac{i}{6}$$

- At $z = e^{5\pi i/6}$, we get

$$(e^{5\pi i/6})^2 \lim_{z \to e^{\pi i/6}} \frac{z - e^{5\pi i/3}}{1 + z^6} = e^{10\pi i/6} \frac{1}{6(e^{5\pi i/6})^5} = e^{10\pi i/6} \frac{e^{-25\pi i/6}}{6} = \frac{e^{-15\pi i/6}}{6}$$

$$= \frac{e^{-3\pi i/6}}{6} = -\frac{i}{6}$$

The first two residues cancel out and we finally get

$$\int_{-\infty}^{\infty} \frac{x^2}{1 + x^6}\, dx = 2\pi i \cdot (-\frac{i}{6}) = \frac{\pi}{3}$$