

# Perspectives in Algebraic Topology

Justin R. Smith

# Topological equivalence

Two spaces  $X$  and  $Y$  are *topologically equivalent* (or homeomorphic) if there exist continuous functions

$$\begin{aligned} f: X &\rightarrow Y \\ g: Y &\rightarrow X \end{aligned}$$

such that  $f \circ g = \mathbf{1}: Y \rightarrow Y$  and  $g \circ f = \mathbf{1}: X \rightarrow X$

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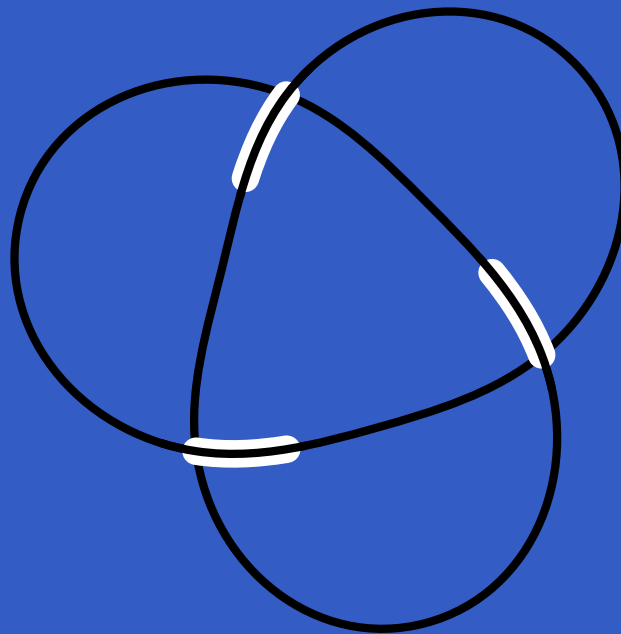
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such that  $f \circ g = \mathbf{1}: Y \rightarrow Y$  and  $g \circ f = \mathbf{1}: X \rightarrow X$

This turns out to be too restrictive to study, except in special cases.

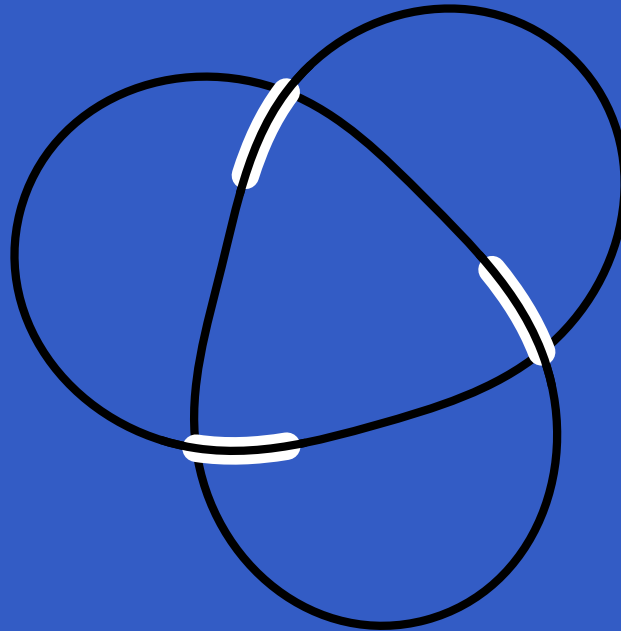
# Example

The coset space of  $\mathbf{SL}_2(\mathbb{Z})$  in  $\mathbf{SL}_2(\mathbb{R})$  is homeomorphic to  $S^3 \setminus K$ , where  $K$  is the trefoil knot:



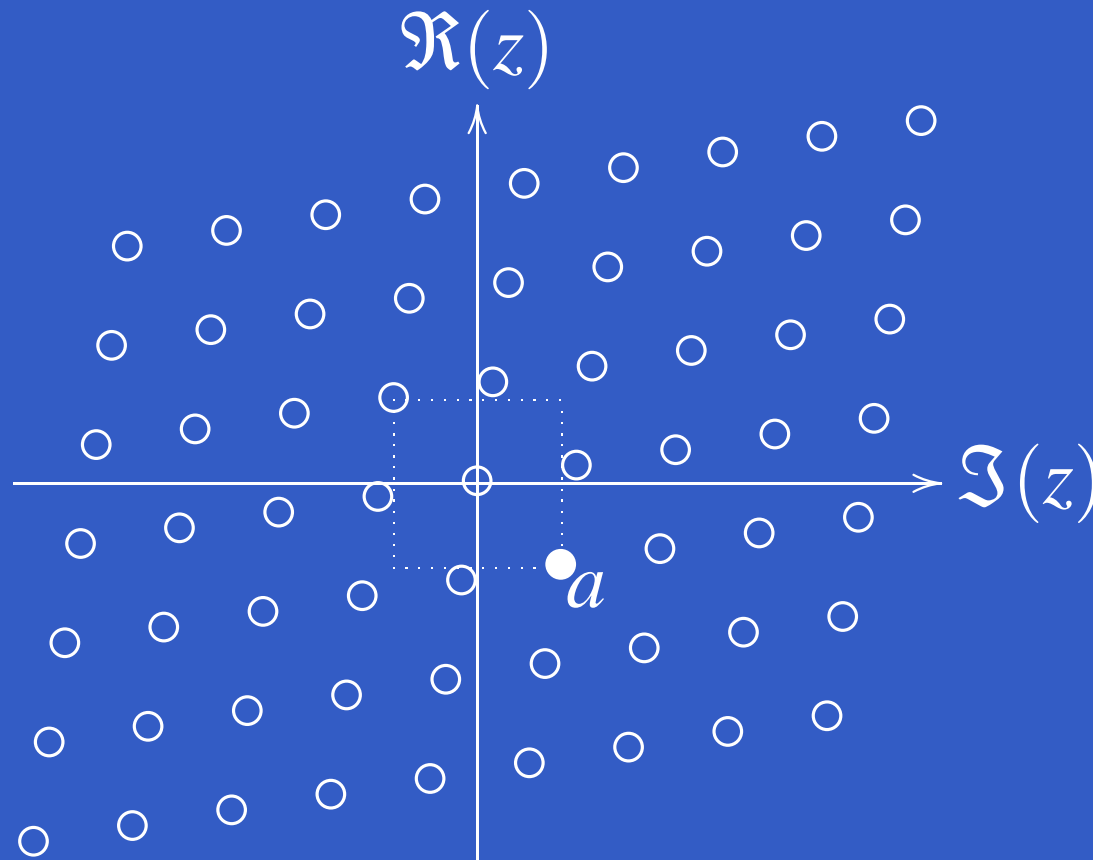
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Each such coset corresponds to an integer lattice in  $\mathbb{C}$ .

# Integer lattice



# Description of the trefoil knot

The trefoil knot is the intersection between the algebraic curve

$$w^2 - z^3 = 0$$

in  $\mathbb{C}^2$  and the unit sphere

$$|w|^2 + |z|^2 = 1$$

# Weierstrass Elliptic functions

Each integer lattice is the period-lattice of a doubly-periodic function satisfying Weierstrass's differential equation

$$\left(\frac{d\wp(z)}{dz}\right)^2 = 4[\wp(z)]^2 - g_2\wp(z) - g_3$$

where

$$g_2 = 60 \sum \frac{1}{\lambda^4}$$

$$g_3 = 140 \sum \frac{1}{\lambda^6}$$

# The correspondence

A pair of numbers  $(g_2, g_3)$  can occur if and only if

$$27g_3^2 - g_2^2 \neq 0$$

This is the discriminant of the polynomial

$$4w^3 - g_2w - g_3$$

# A relationship between maps

Two maps  $f, g: X \rightarrow Y$  are called *homotopic* if there exists a map

$$F: X \times I \rightarrow Y$$

with  $F|X \times \{0\} = f$  and  $F|X \times \{1\} = g$

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You can think of  $F$  as defining a continuous deformation of  $f$  into  $g$ .

# A weaker equivalence relation

Spaces  $X$  and  $Y$  will be called *homotopy equivalent* if there exist maps

$$f: X \rightarrow Y$$

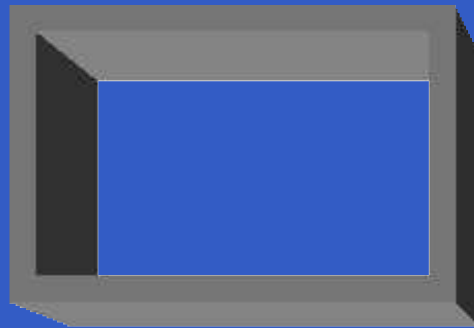
$$g: Y \rightarrow X$$

such that  $f \circ g$  and  $g \circ f$  are *homotopic* to the appropriate identity maps.

# Examples



and



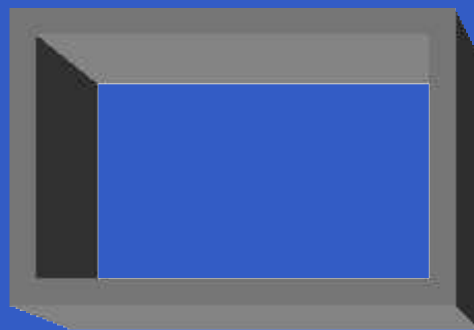
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# Examples



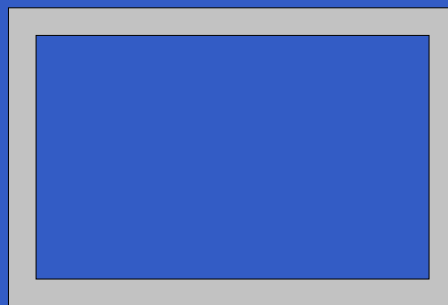
and



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Neither is equivalent to



# Homotopy groups

Given a space,  $X$  with a selected point  $x_0 \in X$ , the  $n^{\text{th}}$  *homotopy group* of  $X$  based at  $x_0$  is defined to be the set of homotopy classes of maps

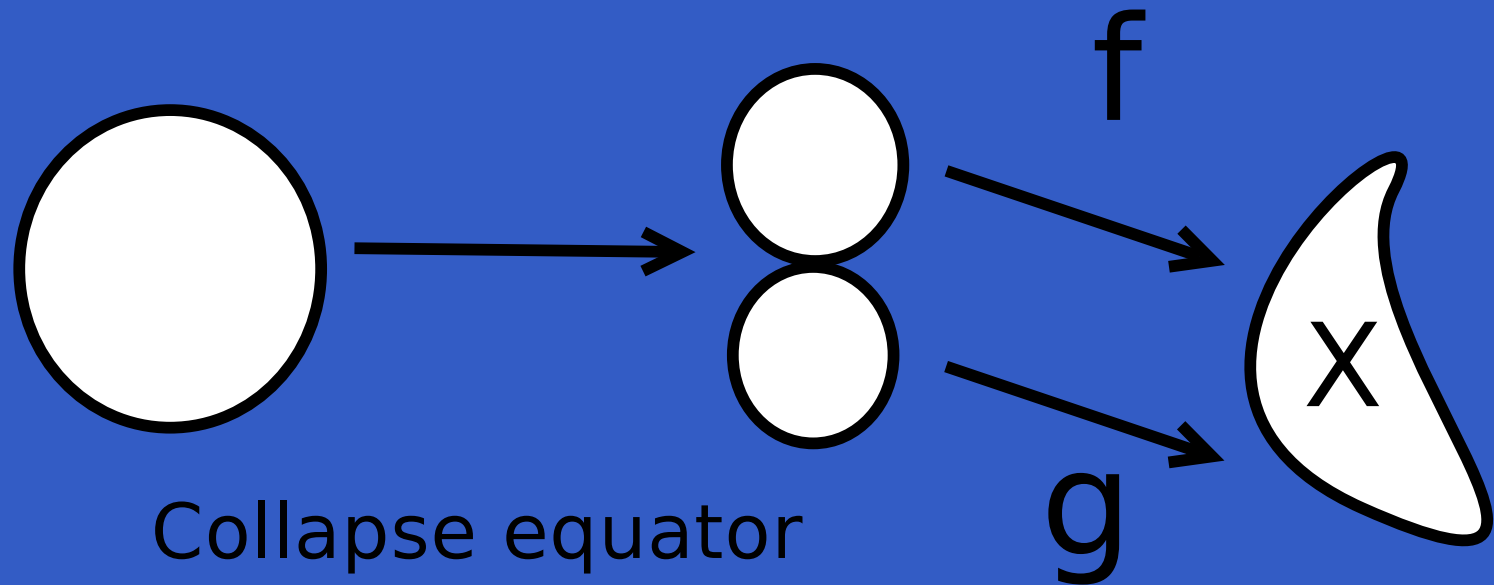
$$S^n \rightarrow X$$

that send a fixed point of  $S^n$  (the north pole, say) to  $x_0$ . It is written

$$\pi_n(X, x_0)$$

# Group operation

Given maps  $f, g: S^n \rightarrow X$  their sum is defined to be the composite



# Induced maps

Any map of spaces

$$f: X \rightarrow Y$$

naturally induces a homomorphism of groups

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Whitehead proved:

**Theorem 0** *The map  $f$  is a homotopy equivalence if and only if these induced maps are isomorphisms for all  $n$ .*

# Some homotopy groups

$$\pi_n(S^n) = \mathbb{Z}$$

and

$$\pi_i(S^n) = \mathbf{0}$$

for  $i < n$ . It was long thought that  $\pi_i(S^n) = \mathbf{0}$  for  $i > n$ .

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Then Hopf showed that

$$\pi_3(S^2) = \mathbb{Z}$$

# The Hopf Map

Regard  $S^3$  as the unit sphere in  $\mathbb{C}^2$ , i.e., pairs of complex numbers  $(z_1, z_2)$  with

$$|z_1|^2 + |z_2|^2 = 1$$

and map the point  $(z_1, z_2) \in S^3$  to the point

$$\frac{z_1}{z_2} \in \mathbb{C} \cup \{\infty\}$$

which is the Riemann sphere,  $S^2$ .

# Homotopy theory is hard

This discovery sparked a massive effort to compute homotopy groups of spheres. Their complexity increases exponentially with dimension. For instance:

$$\pi_{22}(S^4) = \mathbb{Z}_{40} \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_2)^5 \oplus (\mathbb{Z}_3)^2$$

The problem of describing all homotopy groups of spheres (even of  $S^2$ ) is still open.

# Polyhedra

There is one class spaces for which homotopy theory is solved: abelian topological groups. To describe them, it is necessary to describe how to build topological spaces called polyhedra. Their building blocks are *simplices*.

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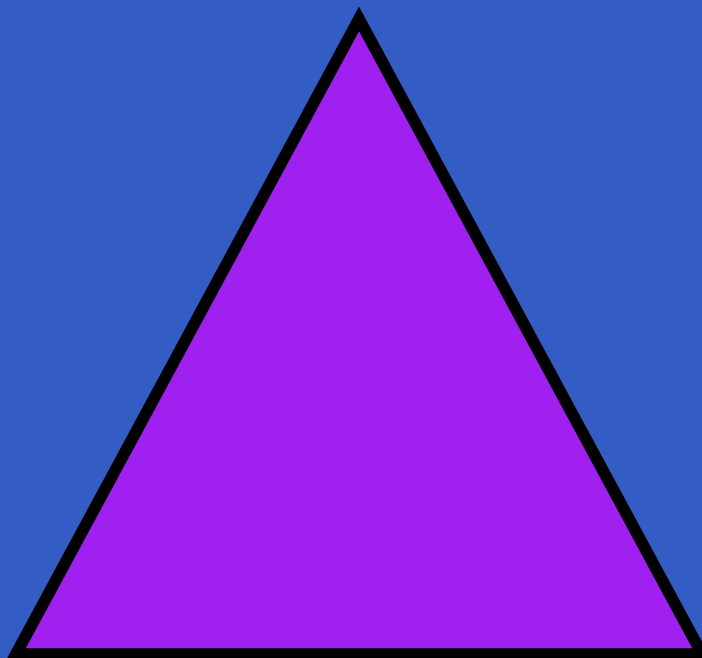
Their building blocks are *simplices*.

The *standard  $n$ -simplex* can be defined as the convex set

$$\{(x_0, \dots, x_n), \sum_{i=0}^n x_i = 1, x_0 \geq 0, \dots, x_n \geq 0\}$$

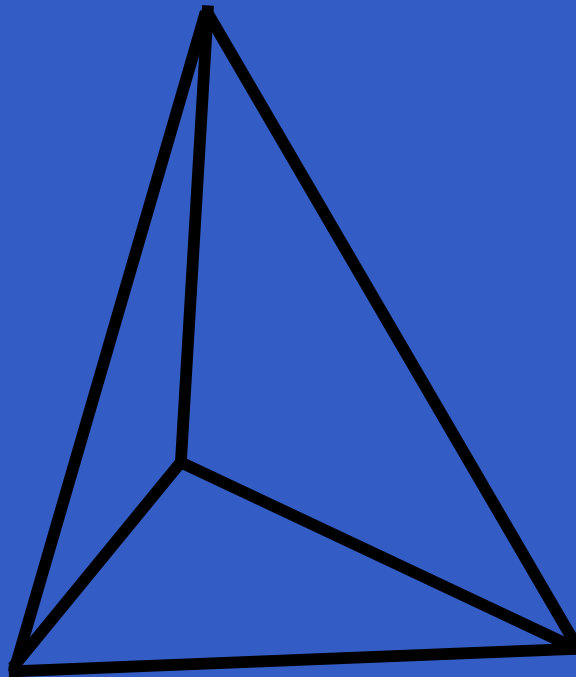
# Examples

A 0-simplex is a point and a 1-simplex is a line-segment. A 2-simplex is a triangle



# 3-simplex

and a 3-simplex is a tetrahedron



# Simplicial sets

The faces of an  $n$ -simplex are  $n - 1$ - simplices and a simplicial set is constructed by

1. gluing together simplices along their common faces.
2. collapsing some simplices onto one of their faces.

# Simplicial abelian groups

When a simplicial set is also an *abelian group*, homotopy theory simplifies. Suppose  $A$  is a simplicial-abelian group — i.e. a simplicial set whose simplices in each dimension form an abelian group. We assume the group-operations respect face-maps (it turns out that this means the space formed from these simplices is a topological abelian group).

# Boundary operations

For an simplex  $\Delta^n \in A$  in a simplicial abelian group, define its boundary by

$$\partial_n \Delta^n = \sum_{i=0}^n (-1)^i F_i \Delta^n$$

Since  $A$  is a simplicial abelian group, these sums and differences have geometric meaning.

# Chain complex

We get a sequence of abelian-group homomorphisms

$$\partial_n: A_n \rightarrow A_{n-1}$$

for all  $n > 0$ , and  $\partial_{n-1} \circ \partial_n = 0$ . If  $x \in A_n$  has the property that  $\partial_n(x) = 0$ , then there exists a map

$$f: S^n \rightarrow A$$

whose image is precisely  $x$  (since  $x$  is a single simplex whose boundary has been collapsed to a point).

# Homotopy groups

On the other hand, if  $x \in A_n$  with  $\partial x = 0$  and  $x = \partial y$ , then there exists a simplex  $y \in A_{n+1}$  whose boundary is  $x$ . This implies that  $x$  represents the  $0$  element of  $\pi_n(A)$ . One can prove

$$\pi_k(A) = \frac{\ker \partial_n: A_n \rightarrow A_{n-1}}{\text{im} \partial_{n+1}: A_{n+1} \rightarrow A_n}$$

# Free abelian groups

Given any simplicial set,  $X$ , one can imbed it in the free-abelian group generated by its simplices

$$X \hookrightarrow \mathbb{Z}X$$

Note that  $\mathbb{Z}X$  has many more simplices than  $X$  — it has one for every *formal linear combination* of simplices of  $X$  .

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One can now define computable homotopy invariants

$$H_k(X) = \pi_k(\mathbb{Z}X)$$

called *homology groups*.

# Rational homotopy theory

Quillen tried to make homotopy theory more tractable by eliminating the torsion. A map

$$f: X \rightarrow Y$$

is a *rational homotopy equivalence* if

$$f^*: \pi_k(X) \otimes \mathbb{Q} \rightarrow \pi_k(Y) \otimes \mathbb{Q}$$

is an isomorphism for all  $k \geq 0$ .

# Quillen's result

Quillen then discovered a coalgebra structure on  $\mathbb{Q}X$  that characterized rational homotopy theory, i.e.

**Theorem 0** *Any if  $\pi_1(X) = \pi_1(Y) = \mathbf{0}$ , algebraic homomorphism*

$$g: \mathbb{Q}X \rightarrow \mathbb{Q}Y$$

*that preserves coalgebra structures, is induced by a map*

$$\hat{g}: X \rightarrow Y$$

# Corollary

**Corollary 0** *Two simplicial sets,  $X$  and  $Y$  with  $\pi_1(X) = \pi_1(Y) = \mathbf{0}$  are rationally homotopy-equivalent if and only if there exists a homology equivalence*

$$g: \mathbb{Q}X \rightarrow \mathbb{Q}Y$$

*preserving coalgebra structures.*

# My work

I hope to extend this to a result over the integers (where most of homotopy theory lives). Problems:

- Coalgebra structures are not enough. One must go to coalgebras over operads, which encode higher homotopy theory (Steedrod operations).
- One must have a homotopy theory of coalgebras over operads — including cofree coalgebras over operads.

# Operads

An operad is a sequence of objects  $\{U_k\}$  together with operations

$$\star_k: U_m \otimes U_n \rightarrow U_{m+n-k}$$

where  $1 \leq k \leq n$ . Here,  $U_k$  is equipped with an action of  $S_k$  (the symmetric group).

Canonical example:

$$\{\mathbf{hom}(C, C^k)\}$$

# Coalgebra over an operad

A coalgebra,  $C$ , over an operad  $\{U_k\}$  is a operad morphism

$$\{U_k\} \rightarrow \{\mathbf{hom}(C, C^k)\}$$

or (equivalently) a sequence of morphisms preserving actions of the symmetric group

$$U_k \otimes C \rightarrow C^k$$

The operad we'll be interested in is the Barratt-Eccles operad

$$\{\mathbf{RS}_k\}$$

# Cosimplicial resolutions

Given a simplicial set  $X$ , we can define its *cosimplicial resolution* as

$$\mathbb{Z}X \xrightarrow{f_0} \mathbb{Z}^2X \xrightarrow{f_1} \dots$$

where  $f_0$  sends  $n \cdot x$  to  $n \cdot \mathbf{1} \cdot x$  and  $f_1$  sends  $n \cdot x$  to  $\mathbf{1} \cdot n \cdot x$ . Note that  $f_0$  is a group morphism and  $f_1$  is not. Also note that  $f_0$  agrees with  $f_1$  only on the image of  $X$  in  $\mathbb{Z}X$ .

# A cosimplicial resolution's total space

Consider the diagram

$$\begin{array}{ccccc}
 \Delta^0 \times \Delta^k & \xrightarrow{F_i \times 1} & \Delta^1 \times \Delta^k & \xrightarrow{F_j \times 1} & \dots \\
 \downarrow & & \downarrow & & \\
 \mathbb{Z}X & \xrightarrow{f_i} & \mathbb{Z}^2 X & \xrightarrow{f'_j} & \dots
 \end{array}$$

The space whose  $k$ -simplices are the vertical (sets of) maps that commute with all horizontal maps is called  $\mathbb{Z}_\infty X$ , the  $\mathbb{Z}$ -completion of  $X$ .

# Properties

- There always exists a natural map

$$h: X \rightarrow \mathbb{Z}_\infty X$$

that induces isomorphisms in homology.

- If  $\pi_1(X) = \mathbf{0}$  then  $h$  is a homotopy equivalence.

# My construction

I construct a space similar to  $\mathbb{Z}_\infty X$  but formed with respect to the cofree coalgebra  $\mathcal{F}^* \mathbb{Z}X$  rather than  $\mathbb{Z}X$ . We get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathbf{h}} & \mathbb{Z}_\infty X \\ & \searrow & \downarrow \\ & & \mathcal{F}^* \mathbb{Z}X \end{array}$$

and a map  $\mathcal{F}^* \mathbb{Z}X \rightarrow \mathbb{Z}_\infty X$

# Coalgebra morphisms to maps

Any coalgebra morphism

$$f: \mathbb{Z}X \rightarrow \mathbb{Z}Y$$

induces a map

$$\mathcal{F}\mathbb{Z}X \rightarrow \mathcal{F}\mathbb{Z}Y$$

that gives rise to a map

$$\mathbb{Z}_\infty X \rightarrow \mathbb{Z}_\infty Y$$